The processes of De Bakker and Zucker represent bisimulation equivalence classes

Rob van Glabbeek & Jan Rutten
Centre for Mathematics and Computer Science
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

The basic fact expressed by the title is not difficult to prove, in a sense is well known, and has yet never been proved in public. Voilà three reasons for this contribution.

DEFINITION 1 (*LTS*): A *labelled transition system* is a triple $\mathcal{Q}=(S,A,\rightarrow)$ consisting of a set of *states* S, a set of *labels* A, and a *transition relation* $\rightarrow \subseteq S\times A\times S$. We shall write $s\xrightarrow{a}s'$ for $(s,a,s')\in\rightarrow$. A LTS is called *image finite* if for all $s\in S$ and $a\in A$ the set $\{s' \colon s\xrightarrow{a}s'\}$ is finite.

DEFINITION 2: Let $\mathcal{Q}=(S,A,\rightarrow)$ be a LTS. A relation $R\subseteq S\times S$ is called *a (strong) bisimulation* if it satisfies for all $s,t\in S$ and $a\in A$:

$$(sRt \wedge s\xrightarrow{a}s') \Rightarrow \exists t'\in S\ [t\xrightarrow{a}t' \wedge s'Rt']\ \text{and}$$

$$(sRt \wedge t\xrightarrow{a}t') \Rightarrow \exists s'\in S\ [s\xrightarrow{a}s' \wedge s'Rt'].$$

Two states are *bisimilar,* notation $s\leftrightarrow t$, if there exists a bisimulation relation R with sRt. The relation $\leftrightarrow$ is again a bisimulation. Note that bisimilarity is an equivalence relation on states.

DEFINITION 3 (Processes): Let the set of processes P be the unique complete metric space that satisfies the following reflexive equation:

$$P \cong \mathcal{P}_{closed}(A\times P).$$

Let d be the metric on P. The metric on $\mathcal{P}_{closed}(A\times P)$ is the Hausdorff metric d_H induced by the following metric on $A\times P$:

$$\bar{d}(\langle a_1,p_1\rangle, \langle a_2,p_2\rangle) = \begin{cases} 1 & \text{if } a_1\neq a_2 \\ \tfrac{1}{2}\cdot d(p_1,p_2) & \text{if } a_1=a_2. \end{cases}$$

The Hausdorff metric d_H is given, for every $X, Y \in \mathcal{P}_{closed}(A \times P)$, by

$$d_H(X,Y) = \max\{\sup_{x \in X}\{d(x,Y)\}, \sup_{y \in Y}\{d(y,X)\}\},$$

where $d(x,Z) = \inf_{z \in Z}\{\bar{d}(x,z)\}$ for every $Z \in \mathcal{P}_{closed}(A \times P)$ and $x \in A \times P$.
(By convention $\sup \varnothing = 0$ and $\inf \varnothing = 1$.)

DEFINITION 4: Let $\mathcal{Q} = (S, A, \rightarrow)$ be an image finite LTS. We define a mapping $\mathfrak{M}: S \rightarrow P$ by

$$\mathfrak{M}[\![s]\!] = \{<a, \mathfrak{M}[\![s']\!]>: s \xrightarrow{a} s'\}.$$

Actually, the precise definition of $\mathfrak{M}$ is $\mathfrak{M}[\![s]\!] = i(\{<a, \mathfrak{M}[\![s']\!]>: s \xrightarrow{a} s'\})$, with $i: \mathcal{P}_{closed}(A \times P) \rightarrow P$ an isometry between $\mathcal{P}_{closed}(A \times P)$ and P, but for convenience we usually leave out isometry symbols. Remark that the isometry i is necessary to stay within well-founded set theory.

We can justify this recursive definition by taking $\mathfrak{M}$ as the unique fixed point (Banach's theorem) of a contraction $\Phi: (S \rightarrow P) \rightarrow (S \rightarrow P)$, defined by

$$\Phi(F)(s) = \{<a, F(s')>: s \xrightarrow{a} s'\}.$$

The fact that Φ is a contraction can be easily proved. The closedness of the set $\Phi(F)(s)$ is an immediate consequence of the image finiteness of $\mathcal{Q}$: Consider a Cauchy sequence $(<a_i, F(s_i)>)_i$ in $\Phi(F)(s)$. From the definition of the metric on $A \times P$ it follows that there exist $\bar{a} \in A$ and $I \in \mathbb{N}$ such that $a_i = \bar{a}$ for all $i > I$. Because $\mathcal{Q}$ is image finite there exists $\bar{s}$ with $s_i = \bar{s}$ for infinitely many i's. Thus the entire sequence $(<a_i, F(s_i)>)_i$ has $<\bar{a}, F(\bar{s})> \in \Phi(F)(s)$ as its limit.

THEOREM 1: *Let $\mathcal{Q} = (S, A, \rightarrow)$ be an image finite LTS. Then:*

$$\forall s, t \in S\ [s \leftrightarrow t \Leftrightarrow \mathfrak{M}[\![s]\!] = \mathfrak{M}[\![t]\!]].$$

PROOF: Let $s, t \in S$.

$\Leftarrow$:

Suppose $\mathfrak{M}[\![s]\!] = \mathfrak{M}[\![t]\!]$. We define a relation $\equiv \subseteq S \times S$ by

$$s' \equiv t' \Leftrightarrow \mathfrak{M}[\![s']\!] = \mathfrak{M}[\![t']\!].$$

From the definition of $\mathfrak{M}$ it is straightforward that $\equiv$ is a bisimulation relation on S: Suppose $s' \equiv t'$ and $s' \xrightarrow{a} s''$; then $<a, \mathfrak{M}[\![s'']\!]> \in \mathfrak{M}[\![s']\!] = \mathfrak{M}[\![t']\!]$; thus there exists $t'' \in S$ with $t' \xrightarrow{a} t''$ and $\mathfrak{M}[\![s'']\!] = \mathfrak{M}[\![t'']\!]$, that is, $s'' \equiv t''$. Symmetrically, the second property of a bisimilation relation holds. From the hypothesis we have $s \equiv t$. Thus we have $s \leftrightarrow t$.

$\Rightarrow$:

Let $R \subseteq S \times S$ be a bisimulation relation with sRt. We define

$$\epsilon = \sup_{s', t' \in S}\{d(\mathfrak{M}[\![s']\!], \mathfrak{M}[\![t']\!]): s'Rt'\}.$$

We prove that $\epsilon = 0$, from which $\mathfrak{M}[\![s]\!] = \mathfrak{M}[\![t]\!]$ follows, by showing that $\epsilon \leq \frac{1}{2} \cdot \epsilon$. We prove for all s', t' with $s'Rt'$ that $d(\mathfrak{M}[\![s']\!], \mathfrak{M}[\![t']\!]) \leq \frac{1}{2} \cdot \epsilon$. Consider $s', t' \in S$

with $s'Rt'$. From the definition of the Hausdorff metric on P it follows that it suffices to show

$$d(x,\mathfrak{M}[\![t']\!]) \leqslant \tfrac{1}{2}\cdot\epsilon \text{ and } d(y,\mathfrak{M}[\![s']\!]) \leqslant \tfrac{1}{2}\cdot\epsilon$$

for all $x\in\mathfrak{M}[\![s']\!]$ and $y\in\mathfrak{M}[\![t']\!]$. We shall only show the first inequality, the second being similar. Consider $<a,\mathfrak{M}[\![s'']\!]>$ in $\mathfrak{M}[\![s']\!]$ with $s'\xrightarrow{a}s''$. Because $s'Rt'$ and $s'\xrightarrow{a}s''$ there exists $t''\in S$ with $t'\xrightarrow{a}t''$ and $s''Rt''$. Therefore

$$\begin{aligned} d(<a,\mathfrak{M}[\![s'']\!]>,\ \mathfrak{M}[\![t']\!]) &= d(<a,\mathfrak{M}[\![s'']\!]>,\ \{<\bar{a},\mathfrak{M}[\![\bar{t}]\!]>: t'\xrightarrow{\bar{a}}\bar{t}\}) \\ &\leqslant [\text{ we have: } d(x,Y)=\inf\{d(x,y): y\in Y\}\ ] \\ &\quad d(<a,\mathfrak{M}[\![s'']\!]>,\ <a,\mathfrak{M}[\![t'']\!]>) \\ &= \tfrac{1}{2}\cdot d(\mathfrak{M}[\![s'']\!],\ \mathfrak{M}[\![t'']\!]) \\ &\leqslant [\text{ because } s''Rt''\ ]\ \tfrac{1}{2}\cdot\epsilon. \quad \square \end{aligned}$$

Next we will generalise theorem 1 to the case that $\mathcal{Q}$ is not required to be image finite. For this purpose we will work in Aczels universe of non-well-founded sets. This universe is an extension of the Von Neuman universe of well-founded sets, where the axiom of foundation (every chain $x_0 \ni x_1 \ni \cdots$ terminates) is dropped. Instead an anti-foundation axiom (AFA) is adopted, saying that systems of equations like the one in definition 4 have unique solutions. Let $\mathcal{V}$ be this universe. In $\mathcal{V}$ there exists a unique complete metric space P satisfying

$$P=\mathcal{P}_{closed}(A\times P).$$

This space can be regarded as a canonical representative of the space from definition 3 in the universe of non-well-founded sets. It can be obtained from any constructed solution of the domain equation in definition 3 by means of projection. Since this canonical representative contains non-well-founded sets indeed, it can not be found in the Von Neuman universe.

We can now extend definition 4 with image infinite LTSs.

DEFINITION 5: Let $\mathcal{Q}=(S,A,\to)$ be a LTS. We define a mapping $\mathfrak{M}:S\to\mathcal{V}$ by

$$\mathfrak{M}[\![s]\!]=\{<a,\mathfrak{M}[\![s']\!]>: s\xrightarrow{a}s'\}.$$

If $\mathcal{Q}$ is not image finite, $\mathfrak{M}[\![s]\!]$ for $s\in S$ may be outside P.

THEOREM 2: *Let FSA* $=(S,A,\to)$ *be a LTS. Then:*

$$\forall s,t\in S\ [s\leftrightarrow t \Leftrightarrow \mathfrak{M}[\![s]\!] = \mathfrak{M}[\![t]\!]].$$

PROOF: This theorem follows immediately from the categorical considerations in Aczels lecture notes on non-well-founded sets. Below we provide a direct non-categorical proof.

$\Leftarrow$:

Exactly as before.

$\Rightarrow$:
Let $\mathfrak{M}^*: S \rightarrow \mathcal{V}$ denote the unique solution of

$$\mathfrak{M}^*[\![s]\!] = \{<a, \mathfrak{M}^*[\![r']\!]>: \exists r \in S\ [r \leftrightarrow s \wedge r \xrightarrow{a} r']\}.$$

As for $\mathfrak{M}$ it follows from AFA that such a unique solution exists. Since $\leftrightarrow$ is an equivalence relation it follows that

$$s \leftrightarrow t \ \Rightarrow\ \mathfrak{M}^*[\![s]\!] = \mathfrak{M}^*[\![t]\!]. \qquad (*)$$

Hence it remains to be proven that $\mathfrak{M}^* = \mathfrak{M}$. This can be done by showing that $\mathfrak{M}^*$ satisfies the equations $\mathfrak{M}[\![s]\!] = \{<a, \mathfrak{M}[\![s']\!]>: s \xrightarrow{a} s'\}$, which have $\mathfrak{M}$ as unique solution. So it has to be established that

$$\mathfrak{M}^*[\![s]\!] = \{<a, \mathfrak{M}^*[\![s']\!]>: s \xrightarrow{a} s'\}.$$

The direction "$\supseteq$" follows directly from the reflexivity of $\leftrightarrow$. For "$\subseteq$", suppose $<a,X> \in \mathfrak{M}^*[\![s]\!]$. Then $\exists r,r'$: $r \leftrightarrow s$, $r \xrightarrow{a} r'$ and $X = \mathfrak{M}^*[\![r']\!]$. Since $\leftrightarrow$ is a bisimulation, $\exists s'$: $s \xrightarrow{a} s'$ and $r' \leftrightarrow s'$. Now from (*) it follows that $X = \mathfrak{M}^*[\![r']\!] = \mathfrak{M}^*[\![s']\!]$. Therefore $<a,X> \in \{<a, \mathfrak{M}^*[\![s']\!]>: s \xrightarrow{a} s'\}$, which had to be established.